Numerical solution of 1st order (differential equation

An ordinary differential equation is one involving a single independent variable These are classified according to the order of derivative terms involved and their power.

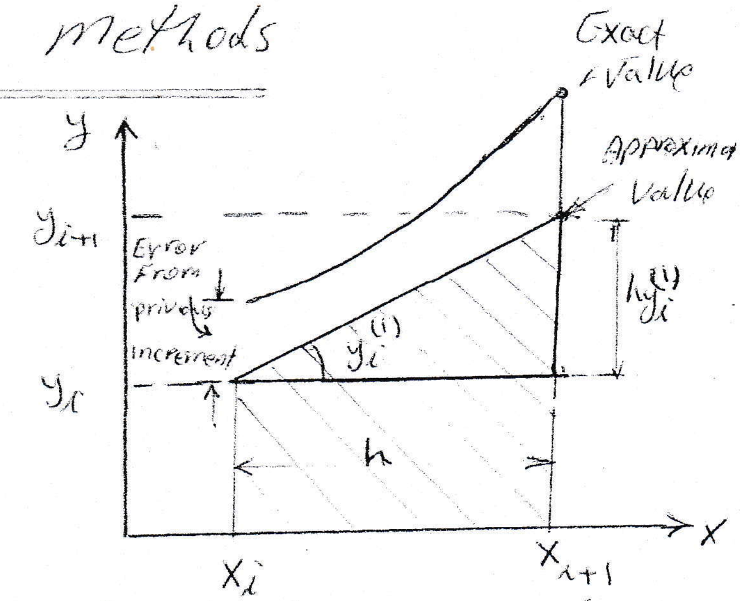
There are many method to solve the ordenary differenti eq. one of this

EULER And EULER modify methods

Let $\frac{dy}{dx} = f(x,y)$

$$\boxed{y_{i+1} = y_i + K_1}$$

where $K_1 = h f(x_i, y_i)$.



Graphical illustration of Euler basic method

EX solve the following differential equation fo $0 \le x \le 1$ using $h = 0.1$ $\frac{dy}{dx} - \frac{1}{2}y = 0$, $y(0) = 1$ using Euler method

sol for $i = 0$, $\boxed{\frac{dy}{dx} = \frac{1}{2}y = f(x_i, y_i)}$, $\boxed{y_{i+1} = y_i + h f(x_i, y_i)}$

$y_1 = y_0 + h f(x_0, y_0) \Rightarrow y_1 = 1 + 0.1 f(0, 1)$

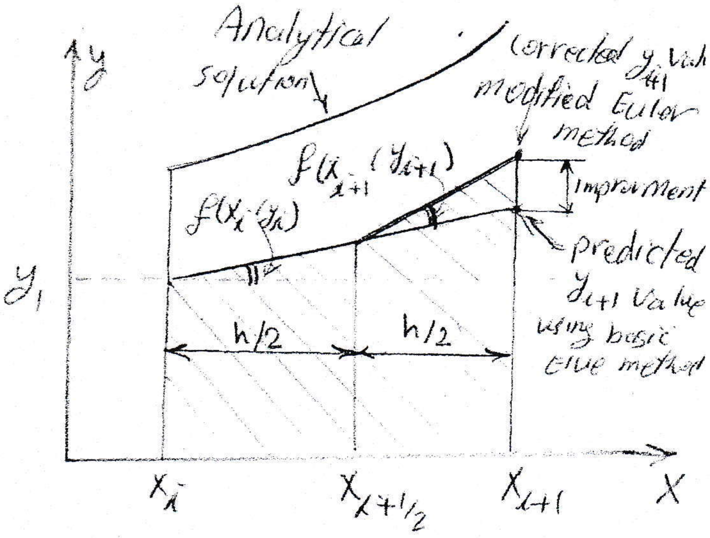
$y_1 = 1 + 0.1(0.5 \times 1) = 1.05$ $\boxed{y_1 = 1.05}$

$i = 1$

$y_2 = y_1 + h f(x_1, y_1) = 1.05 + 0.1(0.5 \times 1.05) = 1.1025$

$i = 2, \dots, 9$ a summary of the results is given in table

i	X_i	y_i	$f(X_i, y_i)$	y_{i+1} EULER	y_{i+1} EXACT
0	0	1.000	0.5	1.05	1.0512711
1	0.1	1.05	0.525	1.1025	1.1051709
2	0.2	1.1025	0.55125	1.157625	1.1618834
3	0.3	1.157625	0.5788125	1.2155063	1.2214027
4	0.4	1.215506	0.6077	1.2762816	1.2840254
5	0.5	1.2762816	0.6381408	1.3400956	1.34985
6	0.6	1.3400956	0.6700478	⋮	⋮
⋮	⋮	⋮	⋮		
9	0.9	1.5513282	0.7756641	1.6288946	1.6487212



Modified Euler Method

$$y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2)$$

$$K_1 = h f(X_i, y_i)$$

$$K_2 = h f(X_{i+1}, y_i + K_1)$$

<u>EX</u> Rework example above using the modified Euler method

for first interval $i = 0$

$$\frac{dy}{dx} = \frac{1}{2} y = f(X_i, y_i)$$

$$K_1 = h f(X_i, y_i) = 0.1 f(X_0, y_0)$$

$$f(0,1) = \frac{1}{2}(1) = 0.5 \qquad \therefore K_1 = 0.1 * (0.5) = 0.05$$

$$K_2 = h f(X_1, y_0 + K_1) = 0.1\left(\frac{1}{2}(1 + 0.05)\right) = 0.0525$$

$$\therefore y_{i+1} = y_i + \frac{1}{2}(K_1 + K_2) = 1 + \frac{1}{2}(0.05 + 0.0525) = 1.05125$$

Numerical Analysis :-

Is the branch of mathmatics that concernin with of methods of obtaining numerical results for many problems. The solution of a problem involving numerical work consists of the following steps:-

1) Modling :-

Formulating the problem in math. terms.

2) Choise of numetrical method and calculate the error, step size, ---

3) Programing using alaugn

4) operation Done by computer

5) Interpretation of results

$$E = \epsilon = \bar{a} - a$$

absolute error — Exact Value — Approximate Value

Solution of the eqs with one Variable

Iteration :-

$f(x) = 0$ —— ① to solve eq. ①

1) Fixed point method
2) Newton-raphson (NR) method.
3) Bisection method.

1) Fixed Point method:

1) Eq. 1 is transform to
$X = g_m(X)$, $m = 1, 2, 3, \ldots$

2) choising an initial Value X_0 and compute a sequence $X_1, X_2, \ldots, X_n$ from the relation
$X_{n+1} = g(X_n)$, $n = 0, 1, 2, \ldots$

3) The solu. depends on the speed of convergence of the sequence and the choise of X_0.

4) The Value of X_0 is choising (chosen) such that
$|g'(x)| < 1$

Ex: Use the fixed point method to solve
$f(x) = X^2 - 3X + 1 = 0$ (Ans: 2.612034 & 0.331966)

$g_1(X) = X = \frac{1}{3}(X^2+1)$

$X_{n+1} = \frac{1}{3}(X_n^2+1)$

$|g_1'(X)| = \frac{2}{3}X < 1$, ∴ let $X_0 = 1$.

$g_2(X) = X = 3 - \frac{1}{X}$

$-x^2 = -3x + 1$

$X_{n+1} = 3 - \frac{1}{X_n}$

$|g_2'(x)| = \frac{1}{x^2}$ (all x except $x = 0$)

let $X_0 = 1$ & 3

n	X_n	X_{n+1}
0	1	0.667
1	0.667	0.481
2	0.481	0.411
3	0.411	0.390
4		0.384
5		0.382

للتأكد نعوض بـ $f(x)$ حيث $f(x) \cong 0$

n	X_n	X_{n+1}
0	1	2
1	2	2.5
2	2.5	2.6
3	2.6	2.615

n	X_n	X_{n+1}
0	3	2.667
1	2.667	2.625
2		2.619
3		2.618

H.W: Use fixed point method to solve

$f(x) = x^2 - 4x + 2 = 0$

Ans:- $X = 2 \pm \sqrt{2}$

Sheet PDE :-

Q.1:

a) $0.02 \sin x$

$u(x,t) = \sum_{n=1}^{\infty} D_n \cos \lambda n t \sin \frac{n\pi}{\ell} x \quad , \ell = \pi \ \& \ \lambda = \frac{cn\pi}{x} = 1$

$D_n = \frac{2}{L} \int_0^{\ell} f(x) \sin \frac{n\pi}{L} x \, dx \ \& \ D_n = \frac{2}{\pi} \int 0.02 \sin x \sin \frac{n\pi}{\ell} x \, dx$

$D_n = \frac{0.02}{\pi} \int_0^{\pi} \cos(x - nx) \, dx - \int_0^{\pi} \cos(x + nx) \, dx$

$D_n = D_1$

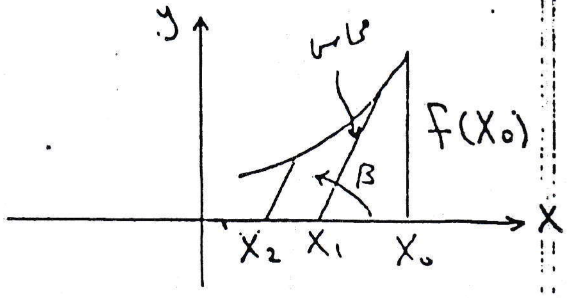
$D_n = \frac{0.02}{\pi} \int_0^{\pi} dx + \frac{1}{2} \sin x \Big]_0^{\pi} = 0$

$= \frac{0.02}{\pi} x \Big]_0^{\pi} = \frac{0.02}{\pi} \pi = 0.02$

$\therefore u(x,t) = 0.02 \cos t \sin x$

Newton Raphson methode :-

It is used for solving $f(x)=0$ where $f(x)$ has a continuous direvative.



$$\tan\beta = \bar{f}(x_0) = \frac{f(x_0)}{x_0 - x_1}$$

$$x_1 = x_0 - \frac{f(x_0)}{\bar{f}(x_0)}$$

in general

قيمة الدالة

القيمة التالية ← $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{\bar{f}(x_n)}}$ $\quad \bar{f}(x_n) \neq 0$, $n = 0, 1, 2, \ldots$

قيمة المشتقة

the advantage of this methode compared with fixed point method :- a) simplicity b) higher speed.

Ex: use N-R to solve $f(x) = x^2 - 3x + 1$

$$\therefore x_{n+1} = x_n - \frac{x_n^2 - 3x_n + 1}{2x_n - 3}, \quad x_0 \neq \frac{3}{2} \text{ because } \bar{f}(x_n) = 0$$

$x_0 = 1$

n	x_n	$f(x_n)$	$\bar{f}(x_n)$	x_{n+1}
0	1	-1	-1	0
1	0	1	-3	0.0333
2	0.333	0.112	-2.334	0.3809
3	0.3809	0.00238	-2.2382	0.3819

→ $f(x)$ تعويض في

* قيمة (X_0) هي اي قيمة ماعدا القيم التي تجعل المشتقة ← 0

Note:- the iteration stops when $f(x) \to 0$

للتأكد من صحة الجذر، نعوض في معادلة $f(x)$

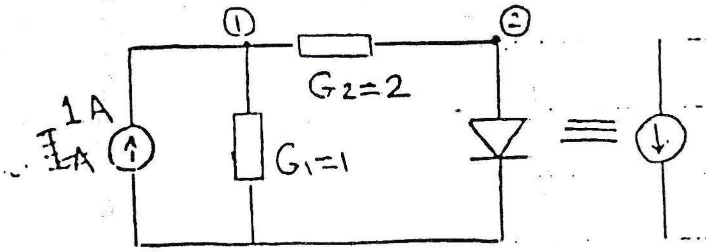
$X_0 = 2$

n	X_n	$f(X_n)$	$\bar{f}(X_n)$	X_{n+1}
0	2	-1	1	3
1	3	1	3	2.667
2	2.667	0.1118	2.334	2.619

H.W:- use N-R to solve $f(x) = x^2 - 4X + 2$

Ans:- $X = 2 \pm \sqrt{2}$

Ex2 Use N-R to find the values of V_1 & V_2 if the diode current $= (e^{40V_2} - 1)$ A, use $V_2(0) = 0.1$ V.



$Y_n V_n = I_n$

$3V_1 - 2V_2 = 1$ ——— ① at node ①

$-2V_1 + 2V_2 + (e^{40V_2} - 1) = 0$ ——— ② at node ②

from ①

$V_1 = \frac{1}{3} + \frac{2}{3} V_2$ ——— ③

نعوض (3) ب (2)

$$\frac{2}{3}V_2 + e^{40V_2} - \frac{5}{3} = 0$$

Let $X = V_2 \Rightarrow f(X) = \frac{2}{3}X + e^{40X} - \frac{5}{3}$

$$X_{n+1} = X_n - \frac{\frac{2}{3}X_n + e^{40X_n} - \frac{5}{3}}{\frac{2}{3} + 40e^{40X_n}}$$

n	X_n	$f(X_n)$	$f'(X_n)$	X_{n+1}
0	0.1	52.99	2184.5	0.07574
1	0.07574	19.07	828.22	0.052712
2	0.052712	6.6	330.09	0.032705
⋮				
6	0.012654	6.7×10^{-4}	67.02	0.012644
7	0.012644	7.9×10^{-6}	66.99	0.012644

$\therefore V_2 = 0.12644 \Rightarrow$ Diode off لأنه أقل من 0.7

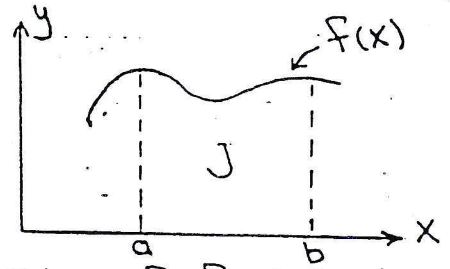
$V_1 = 3.4176 \times 10^{-1}$ نعوض ب (3)

H.W: $V_2(0) = 0.05$ اعادة حل نفس السؤال

Numerical Integration :-

- Is the numerical evaluation of a definite Integral

$$J = \int_a^b f(x)\,dx$$



- (J) is the area under the curve of f(x) between a & b. numerical integration methodes are :-

1) Trapezoidal Rule. قانون شبه المنحرف
2) Simpson's Rule.

1) Trapezoidal Rule :-

قاعدة شبه المنحرف

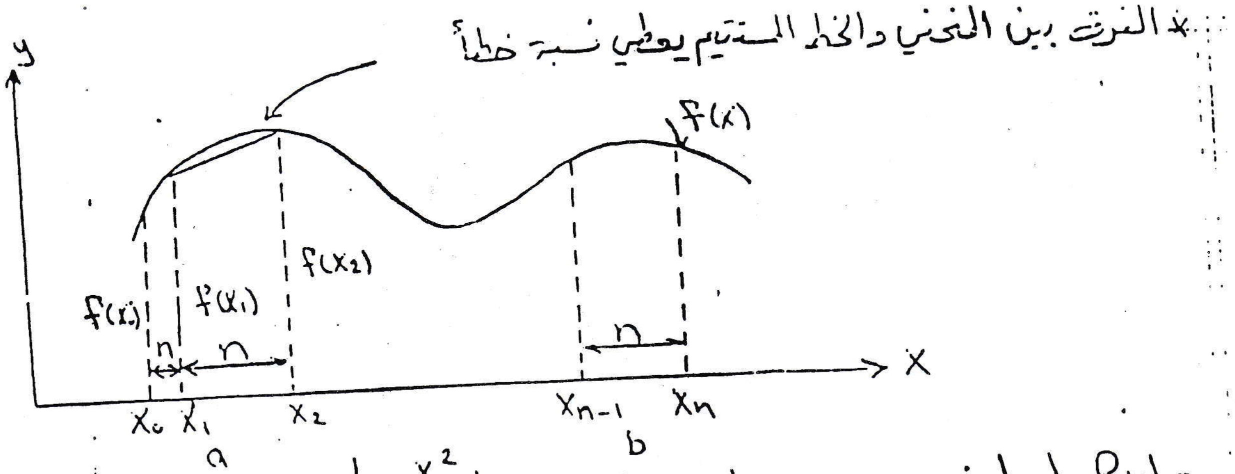
1- we subdivide the interval of Integration into (n) equal subintervals of lengths $\left(h = \frac{b-a}{n}\right)$.

2- approximate f(x) in each sub interval by: biecewise linear function, we obtaine the trapezoidal Rule.

$$J = \int_a^b f(x)\,dx \simeq h\left\{\frac{1}{2} f(\underset{a}{x_0}) + f(x_1) + \cdots\cdots \frac{1}{2} f(\underset{b}{x_n})\right\}$$

لأنها مقترنة بين الفترة التي قبلها والتي تليها

or : $\simeq \frac{h}{2}\left\{f(\underset{a}{x_0}) + 2f(x_1) + 2f(x_2) + \cdots\cdots + f(\underset{b}{x_n})\right\}$



Ex 2 Evalute $J=\int_0^1 e^{-x^2}dx$ using trapezoidal Rule.

$n=10 \rightarrow$ عدد المناطق

solution:-

$$h=\frac{b-a}{n}=\frac{1-0}{10}=0.1$$

$$x_j = x_0 + jh \quad , \quad J=1,2,\dots \qquad f(x_j)=e^{-x_j^2}$$

J	x_i	$f(x_j)=e^{-x_j^2}$
0	0	1 ← f(a)
1	0.1 ↓ الزيادة بين x x = 0.1	0.99005
2	0.2	0.960789
3	0.3	0.913931
4	0.4	0.852144
5	0.5	0.778801
6	0.6	0.697676
7	0.7	0.612626
8	0.8	0.527292
9	0.9	0.444858
10	1	0.367879 ← f(b)

$$J \simeq 0.1 \left\{ \frac{1}{2}(1) + 0.99005 + \cdots + \frac{1}{2}(0.367879) \right\}$$
$$\simeq 0.746211$$

* Error bound :- ‏حساب الـ Error Rang

$$k f''(a) \leqslant E \leqslant k f''(b)$$
$$k = (b-a)^3 / 12n^2$$

بالنسبة للسؤال اعلاه

$$f''(x) = 2(2x^2-1)\, e^{-x^2}$$
$$f''(a) = f''(0) = -2 \quad , \quad f''(b) = f''(1) = 0.73575$$
$$k = 1/1200$$
$$-0.001667 \leqslant E \leqslant 0.00614$$

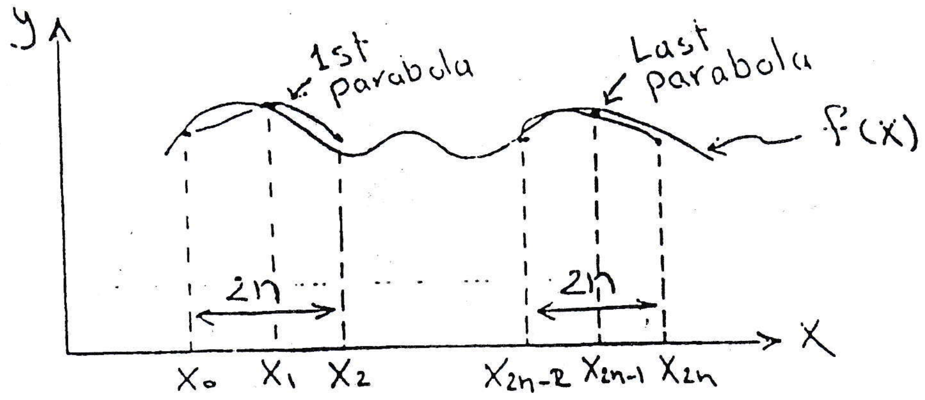
«لزيادة دقة نسبة الخطأ نزيد مناطق التقسيم»

2) Simpson's Rule :-

1- we subdivide the interval of integration $a \leqslant x \leqslant b$ into an even number $(2n)$ of equal sub interval.

$$h = \frac{b-a}{2n}$$

2- we approximate $f(x)$ using biecewise qua-drate approximation in the interval $x_0 \leqslant x \leqslant x_2 = x_0 + 2n$ by Lagrange bolunomial (متعدد الحدود)



$$J \cong \frac{h}{3}\left[G_0 + 4G_1 + 2G_2\right]$$

$G_0 = f(a) + f(b)$

$G_1 = f(x_1) + f(x_3) + \cdots$ الحدود الفردية

$G_2 = f(x_2) + f(x_4) + \cdots$ الحدود الزوجية

Ex: Evaluate $J = \int_0^1 e^{-x^2}\,dx$ using simpsons rule

$2n = 10$

$$h = \frac{b-a}{2n} = \frac{1-0}{10} = 0.1$$

$x_j = x_0 + jh \quad , j = 1, 2, \cdots$

j	x_j	$f(x_j) = e^{-x_j^2}$	
0	0	1 ← f(a)	
1	0.1	0.99005	الحدود الفردية
2	0.2	0.960789	
3	0.3	0.913931	
4	0.4	0.852144	
5	0.5	0.778801	

6	0.6	0.697676
7	0.7	0.612626
8	0.8	0.527292
9	0.9	0.444858
10	1	0.367879 ← f(b)

$$J \cong \frac{0.1}{3}\left\{ \overbrace{1.367879}^{f(a)+f(b)} + 4\underbrace{(3.740266)}_{\text{حدود فردية}} + 2\underbrace{(3.03791)}_{\text{حدود زوجية}} \right\}$$

$$\cong 0.746825$$

Error bound :-

الخطأ

$$Cf^4(a) \leqslant E \leqslant Cf^4(b)$$

$$C = (b-a)^5/180(2n)^4$$

بالنسبة للسؤال اعلاه :-

$$f^4(X) = 4(4X^4 - 12X^2 + 3)e^{-X^2}$$

$$f^4(a) = f^4(0) = 12 \ , \ f^4(b) = f^4(1) = -7.359$$

$$-4 \times 10^{-6} \leqslant E \leqslant 6 \times 10^{-6}$$

تعتبر ادق من الطريقة السابقة

H.W: اعادة حل نفس السؤال وذلك

a) using trapezoidal rule, $n=16$

b) using simpson's rule $2n=16$

Systems of Linear equations:-

Solution using direct & indirect (iterative) methodes:-

تكراري

A system of (n) linear equations in (n) unknowns (variable)

* عدد المعادلات = عدد المجاهيل

$X_1, X_2, \ldots, X_n$

is a set of equations of the form

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = b_1$$
$$a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = b_2$$
$$\vdots$$
$$a_{n1}X_1 + a_{n2}X_2 + \cdots + a_{nn}X_n = b_n$$

two methods are used to solve such equations

1) direct methods :-

a- using (المصفوفات), ($AX=b$)

b- using (Carmers rule).

c- using (gaussian elemenation methode)

2) Indirect method (aterative method) :-

a- Gauss-Seidal (G-S) iteration methode.

b- Jacobi methode.

These methode converges, if

$$\sum_{j=1}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \leq 1 \quad \text{for at least on eq.} \quad i \neq j$$

1) Gauss_Seidal (G-S) iteration method:-

Iteration method are used:-

1) In problem for which convergence is raped نجاني

2) For system of large order but with many zero coefficient (spares system)

$$X_i^{(k)} = \frac{-\sum_{j=1}^{i-1}(a_{ij}\,x_j^{(k)}) - \sum_{j=i+1}^{n}(a_{ij}\,x_j^{(k-1)}) + b_i}{a_{ii}}$$

$i = 1, 2, \dots, n \;, \; i \neq j$

$a_{ii} \neq 0 \;, \; k = 1, 2, \dots$ no. of iterations

Ex: Solve using G-S method:-

$$10X_1 - X_2 + 2X_3 = 6$$
$$-X_1 + 11X_2 - X_3 + 3X_4 = 25$$
$$2X_1 - X_2 + 10X_3 - X_4 = -11$$
$$3X_2 - X_3 + 8X_4 = 15$$

assume $X^{(0)} = (0000)$

$$\sum_{\substack{j=1 \\ i\neq j}}^{n} \left|\frac{a_{ij}}{a_{ii}}\right| \leqslant 1$$ for at least one eq.

Sol.:-

$$X_1^{(k)} = \frac{1}{10} X_2^{(k-1)} - \frac{1}{5} X_3^{(k-1)} + \frac{3}{5}$$

$$X_2^{(k)} = \frac{1}{11} X_1^{(k)} + \frac{1}{11} X_3^{(k-1)} - \frac{3}{11} X_4^{(k-1)} + \frac{25}{11}$$

$$X_3^{(k)} = \frac{-1}{5} X_1^{(k)} + \frac{1}{10} X_2^{(k)} + \frac{1}{10} X_4^{(k-1)} - \frac{11}{10}$$

$$X_4^{(k)} = \frac{-3}{8} X_2^{(k)} + \frac{1}{8} X_3^{(k)} + \frac{15}{8}$$

k	0	1	2	3	4	5 (no of iterations)
$X_1^{(k)}$	0	0.6	1.03	1.0065	1.0009	1.0001
$X_2^{(k)}$	0	2.3272	2.037	2.0036	2.0003	2.0
$X_3^{(k)}$	0	-0.9873	-1.014	-1.0025	-1.0003	-1
$X_4^{(k)}$	0	0.8789	0.9844	0.9983	0.9999	1

$$\therefore X^{(5)} = (\underset{X_1}{1} \quad \underset{X_2}{2} \quad \underset{X_3}{-1} \quad \underset{X_4}{1})$$ للتأكد عوض في المعادلة الاصلية

2) Jacobi method :-

Similar to G-S method but it differs in :-

1) not using improved value and tell a step has been completed.

2) the solution is converge in more iteration than (G-S) method.

$$X_i^{(k)} = \frac{\sum_{j=1}^{n} \{a_{ij}\, x_j^{(k-1)}\} + b_i}{a_{ii}} \quad , \quad i = 1, 2, \dots, n, \ i \neq j$$

$a_{ii} \neq 0$, $k = 1, 2, \dots$ no of iterations.

Ex :- Solve using Jacobi method :-

$$10X_1 - X_2 + 2X_3 = 6$$

$$-X_1 + 11X_2 - X_3 + 3X_4 = 25$$

$\downarrow$

$$2X_1 - X_2 + 10X_3 - X_4 = -11$$
$$3X_2 - X_3 + 8X_4 = 15$$

assume $X^{(0)} = (0000)$

$$\sum_{\substack{j=1 \\ i \neq j}}^{n} \left| \frac{a_{ij}}{a_{ii}} \right| \leqslant 1 \text{ for at least one eq.}$$

Solution :-

$$X_1^{(k)} = \frac{1}{10} X_2^{(k-1)} - \frac{1}{5} X_3^{(k-1)} + \frac{3}{5}$$

$$X_2^{(k)} = \frac{1}{11} X_1^{(k-1)} + \frac{1}{11} X_3^{(k-1)} - \frac{3}{11} X_4^{(k-1)} + \frac{25}{11}$$

$$X_3^{(k)} = \frac{-1}{5} X_1^{(k-1)} + \frac{1}{10} X_2^{(k-1)} + \frac{1}{10} X_4^{(k-1)} - \frac{11}{10}$$

$$X_4^{(k)} = \frac{-3}{8} X_2^{(k-1)} + \frac{1}{8} X_3^{(k-1)} + \frac{15}{8}$$

k	$X_1^{(k)}$	$X_2^{(k)}$	$X_3^{(k)}$	$X_4^{(k)}$
0	0	0	0	0
1	0.6	2.2727	-1.1	1.875
2	1.0473	1.7159	-0.8052	0.8852
↓ 9	0.9997	2.004	-1.0004	1.0006
10	1.0061	1.9988	-0.9998	0.9998

$$X^{(10)} = (1 \quad 2 \quad -1 \quad 1)$$
$$\quad x_1 \quad X_2 \quad X_3 \quad X_4$$